Coupling of the eXtended Finite Element Method and the matching asymptotic development in the modelling of brazed assembly

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ABSTRACT: Modelling of the displacement field of a brazed assembly taking account the presence of a brazed joint which is considered as a singularity is proposed. The model is based on the eXtended Finite Element Method (X-FEM) coupled with the matching asymptotic development (DAR). The fundamentals of our approach is given and illustrated in a 1D example elastic deformations. The DAR solution of that problem is given, as well as the expressions of the DAR inspired enriched X-FEM functions. The implementation in the X-FEM framework is detailed and a case study result is shown and compared with fine grid FEM calculation.

Key words: X-FEM, eXtended Finite Element Method, matching asymptotic development, DAR, brazed joint

1 INTRODUCTION

In the modelling of the behavior of brazed assemblies taking account the presence of a brazed joint, difficulties are usually encountered because of its singularity in the whole assembly. Most of the time, the brazed joint is either ignored or modelled with an extra fine mesh in the zone around brazed joint. In the latter case, computational time can become tremendous and prohibitive. Our audacious idea to couple the eXtended Finite Element Method (X-FEM) and the matching asymptotic development (DAR) proposes to overcome these difficulties.

The X-FEM is based on the idea that the approximation of standard finite element can be extended by the use of enriched functions. Concretely, within the X-FEM, the elements that incorporate a section of a singularity can be attributed an enriched interpolation function, this function being chosen to represent as closely as possible the behaviour of the singularity. The X-FEM has been very successfully used for cracks, with specific functions derived from the fracture mechanics [1]. In addition, this method also shows advantages in the combination with the level set method to model the problems of implicit interface [2], holes or inclusions [3]. In our case of brazed joint, we need to identify a new set of interpolation function. Our new idea is to seek those interpolation functions using the DAR method. It provides the modelling of the behavior of a singular piece of material at two scales with a criteria to match them together : macroscopic (or exterior solution) and microscopic (or interior solution in the "boundary layer") [4]. Starting from the problem of fluid mechanic, DAR is now successfully used in many mechanical sectors, such as: composite materials [5], assemblies of materials [6]…

In this piece of work, the exterior solution of the DAR for the brazed assembly is envisaged as the enriched interpolation function in the stage of preprocessing within the framework of the X-FEM.The present paper details the DAR exterior and interior solutions in a 1D geometry. Then in a second part the integration within the X-FEM framework is explained. The paper finishes with one application.

2 PRESENTATION OF THE REVIEWED MODEL AND THE PRINCIPLE OF DAR AND X-FEM

2.1 Model 1D of the brazed assembly

The brazed assembly is made of two sheets of base materials 1 and 3 which are assembled using the filler metal 2. In our study, we only consider the case 1D of this model which is shown in figure 1.

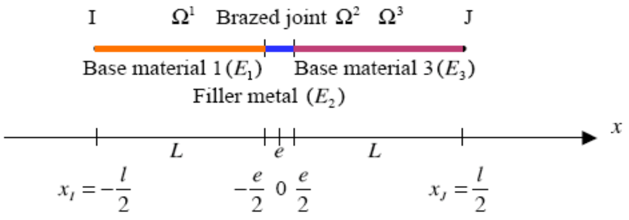

Fig. 1. Model 1D of the brazed assembly

2.2 Approach of the DAR method

Because of its small thickness compared to those of the different base materials, the brazed joint is considered as a singularity. We introduce the ratio $\varepsilon = e / L$ where e and L denote the thicknesses of brazed joint and of sheet of base material respectively. DAR allows us to find the solution of the problem in the form of two asymptotic developments of the small parameter ε. The first development, exterior, provides us with the behaviour in the zone around the brazed joint. The exterior field of displacement is given in $\Omega^1(-)$ and $\Omega^3(+)$ respectively in the form of eq. (1):

$$u_{\mp}^{\varepsilon}(x) = u_{\mp}^{0}(x) + \varepsilon u_{\mp}^{1}(x) \tag{1}$$

Where $u_{\mp}^{0}, u_{\mp}^{1}$ denote the two first exterior terms of DAR. $u_{\mp}^{0}$ characterizes the solution of the unperturbed problem (without Ω^2). However, $\varepsilon u_{\mp}^{1}$ stands for the correction taking into account the brazed joint.

The second development, interior, is valid inside the boundary layer. The brazed joint is zoomed in when applying a change of variable $y = x / \varepsilon$. The zone of the brazed joint is divided into two parts:

$$\begin{cases} \Omega_{-}^{2} : -e/2 \le x \le 0 \Leftrightarrow -L/2 \le y \le 0 \\ \Omega_{+}^{2} : 0 \le x \le e/2 \Leftrightarrow 0 \le y \le L/2 \end{cases} \tag{2}$$

The interior expansion in these two parts is expressed in the form:

$$v_{\mp}^{\varepsilon}(y) = v^{0}(y) + \varepsilon v_{\mp}^{1}(y) \tag{3}$$

Where $v^0, v_{\mp}^{1}$ denote the two first interior terms of the DAR for the two domains $\Omega_{-}^{2}(v_{-}^{1})$ and $\Omega_{+}^{2}(v_{+}^{1})$. The two developments are matched with the following condition at their respective limits:

$$\lim_{x \to 0} u^{\varepsilon} = \lim_{y \to \pm\infty} v^{\varepsilon} \tag{4}$$

Each term of the two developments is determined with the simultaneous utilization of the matching condition above and the classical equations of the model (equilibrium equation, constitutive law, continuity condition) and we found the following solution according to (6) development.

$$\begin{aligned}
u_{-}^{0} &= \frac{E_3(u_J - u_I)}{E_1 x_J - E_3 x_I} x + \frac{E_1 x_J u_I - E_3 x_I u_J}{E_1 x_J - E_3 x_I} \\
u_{+}^{0} &= \frac{E_1(u_J - u_I)}{E_1 x_J - E_3 x_I} x + \frac{E_1 x_J u_I - E_3 x_I u_J}{E_1 x_J - E_3 x_I} \\
v^{0} &= \frac{u_I x_J - u_J x_I}{x_J - x_I} \\
\varepsilon u_{-}^{1} &= \frac{e E_3 (u_I - u_J)(2E_1E_3 - E_2E_3 - E_1E_2)(x - x_I)}{2E_2 (E_1 x_J - E_3 x_I)^2} \\
\varepsilon u_{+}^{1} &= \frac{e E_1 (u_I - u_J)(2E_1E_3 - E_2E_3 - E_1E_2)(x - x_J)}{2E_2 (E_1 x_J - E_3 x_I)^2} \\
\varepsilon v_{-}^{1} &= \frac{e E_3 (u_J - u_I)}{2E_2 (E_1 x_J - E_3 x_I)} [\frac{2E_1 x}{e} + (E_1 - E_2) + \\
&\quad + \frac{x_I (2E_1E_3 - E_2E_3 - E_1E_2)}{(E_1 x_J - E_3 x_I)}] \\
\varepsilon v_{+}^{1} &= \frac{e E_1 (u_J - u_I)}{2E_2 (E_1 x_J - E_3 x_I)} [\frac{2E_3 x}{e} + (E_2 - E_3) + \\
&\quad + \frac{x_J (2E_1E_3 - E_2E_3 - E_1E_2)}{(E_1 x_J - E_3 x_I)}]
\end{aligned} \tag{5}$$

where

x_I, x_J = nodal coordinates

u_I, u_J = nodal displacements

E_1, E_2, E_3 = Young's modulus.

3 COUPLING OF DAR AND X-FEM

Within the X-FEM framework, the field of displacement of the domain that contains a singularity is decomposed into two parts: a standard of finite element approximation (linear) and enriched functions to account for the specific singular behaviour. It usually takes the form of eq. (6).

$$u^h(x) = \sum_{I \in N^{total}} N_I(x) u_I + \sum_{J \in N^{enr}} N_J(x) \psi_J(x) b_J \tag{6}$$

where N_I, N_J = shape functions, ψ_J = enriched function, b_J = additional degree of freedom.

N^{total} = set of all nodes of the domain, N^{enr} = set of nodes whose shape functions support contains the singularity.

The choice of the enriched functions plays the primary role because it determines the relevance of this method. In our study, these functions will be chosen to represent as closely as possible the behaviour of the brazed joint in the global assembly. Within DAR, we obtain two solutions (exterior and interior) which express analytically this behavior.It makes perfect sense to us to combine the two methods. Subsequently, we compared eq. (6) with eq. (1) and (3) in the spirit of identifying possible enriched functions. To achieve this goal we had to keep in mind a few remarks:

1. The solution of the unperturbed problem u^0 of DAR is equivalent to the standard classical part of the X-FEM.
2. The corrected terms $\varepsilon u_{\mp}^{1}$ of the DAR play the role of the enriched part $N_J(x)\psi_J(x)b_J$ in the formulation of the X-FEM. In other words, the perturbation in the meaning of DAR is attributed to the enrichment within the principle of the X-FEM.
Lastly the possible enriched functions had to ensure the fundamental principle of FEM, which are summarised here :

- ensuring the continuity of displacement between two adjacent elements
- canceling in the non-enriched elements
- avoiding an abrupt change between the enriched and unenriched elements through a blending element.

From all these considerations, we arrived to the following formulation of the enriched functions, eq. (7) and (8).
The exterior enriched functions are given by:

$$\psi_I^{ext} = \frac{eE_3 a}{E_2 b}.\frac{(x-x_I)(x_J-x_I)}{x_J-x}$$
$$\psi_J^{ext} = \frac{eE_1 a}{E_2 b}.\frac{(x_J-x)(x_J-x_I)}{x-x_I} \quad (7)$$

The interior enriched functions are expressed by:

$$\psi_I^{\text{int}} = \frac{eE_3}{E_2 b}[\frac{2E_1 x}{e} + (E_1-E_2) + x_I a]\frac{x-x_J}{x_J-x_I}$$
$$\psi_J^{\text{int}} = \frac{eE_1}{E_2 b}[\frac{2E_3 x}{e} + (E_2-E_3) + x_J a]\frac{x-x_I}{x_J-x_I} \quad (8)$$

where the constants are expressed as below:

$$a = \frac{2E_1E_3 - E_2E_3 - E_1E_2}{E_1 x_J - E_3 x_I}$$
$$b = E_1 x_J - E_3 x_I \quad (9)$$

4 X-FEM FORMULATION

4.1 *Weak form*

The weak form of boundary value problem consists of finding $u \in U | \forall v \in U_0$:

$$\int_\Omega \sigma(u):\varepsilon(v)d\Omega = \int_\Omega b.v d\Omega + \int_{\Gamma_t} t.v d\Gamma \quad (10)$$

With σ = Cauchy's stress, ε = strain, b = body force per unit volume, $\bar{u}, t$ = prescribed displacement and stress vector on Γ_u and Γ_t respectively ;
And where
U = space of admissible displacement field
$U = \{u | u \text{ piecewise smooth and } u = \bar{u} \text{ on } \Gamma_u\}$
U = space of test function
$U_0 = \{v | v \text{ piecewise smooth and } v = 0 \text{ on } \Gamma_u\}$
Γ = boundary of Ω
$\Gamma = \Gamma_t \cup \Gamma_u$
By applying the Hooke's law and the relationship between strain and displacement, we obtain the governing equation as eq. (11) :

$$K.u = f \quad (11)$$

where u = displacement vector, f = matrix of exterior force, K = matrix of rigidity given by eq. (12).

$$K = \int_\Omega {}^T BCBd\Omega \quad (12)$$

where B = standard strain-displacement matrix, C = Hooke's tensor.

4.2 *Discretization*

From the form of displacement field of the X-FEM (6) associated to the weak form (10) and the governing equation (11), the discretization can be formulated.
The element approximation of displacement field for element *(IJ)* is given by eq. (13) which can be synthetised in the matrix form eq. (14)

$$u^{(IJ)}(x) = \sum_{i=I,J} N_i(x)u_i + \sum_{i=I,J} N_i(x)\psi_i(x)b_i \quad (13)$$
$$\{u\}^{(IJ)} = [N]^{(IJ)}.\{q\}^{(IJ)} \quad (14)$$

where
$[N]^{(IJ)}$ = matrix of generalized shape functions
$\{q\}^{(IJ)}$ = vector of generalized nodal displacements
In our case, beside the FEM standard elements, we are interested in two types of elements: enriched element and blending element.
The enriched element contains the brazed joint. Thus, all its two nodes are enriched and the matrix writes with components given in eq. (15). The blending elements are adjacent to the enriched element. Only one of its two nodes is enriched and thus its matrix expression uses the components given in eq. (16).

$$[N]^{(IJ)} = [N_I(x) \quad N_I(x)\psi_I(x) \quad N_J(x) \quad N_J(x)\psi_J(x)]$$
$$\{q\}^{(IJ)} = \{u_I \quad b_I \quad u_J \quad b_J\}^T \quad (15)$$

where ψ_I, ψ_J can be taken the values of $\psi_I^{ext}, \psi_J^{ext}$ eq. (7) for the domain outside.

$$[N]^{(IJ)} = [N_I(x) \quad 0 \quad N_J(x) \quad N_J(x)\psi_J(x)]$$
$$\{q\}^{(IJ)} = \{u_I \quad 0 \quad u_J \quad b_J\}^T \quad (16)$$

where ψ_J must be taken the value of ψ_J^{ext} eq. (8).

5 NUMERICAL EXAMPLE

Let us consider the assembly of two sheets of steel ($E_1 = 200000\,MPa;\ E_3 = 160000\,MPa$) brazed together with a copper-silver alloy ($E_2 = 74000\,MPa$). The thickness of the brazing joint is $e = 0.1mm$.
The whole domain is discretized into five elements of constant length $l = 12.3mm$ as depicted in figure 2. We chose to take into account the dimensionless parameter $\varepsilon = 0.0164 = 2e/l$ for the enriched and blended elements. The boundary condition at the node 1 is embedded, while at the node 6, the exterior force $F = 100N$ is applied.

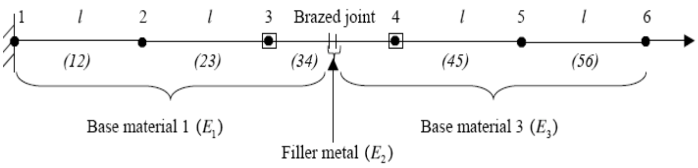

Fig. 2. Model X-FEM 1D of the brazed assemby

The coupling X-FEM-DAR is implemented in MATLAB. An identical model of ABAQUS with a refinement of the mesh in the brazed joint and an analytical model of ressorts are considered as the reference to evaluate the convergence of this method. The results are presented in figures 3 and 4.

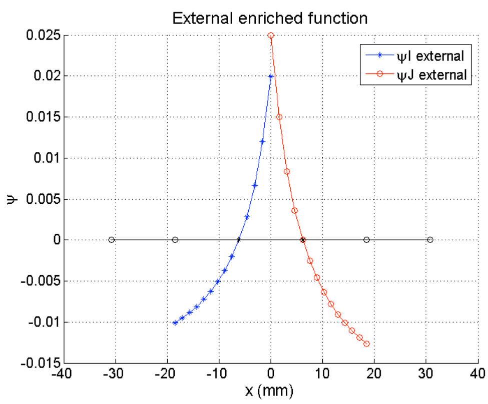

Fig. 3. Representation of the exterior enriched functions

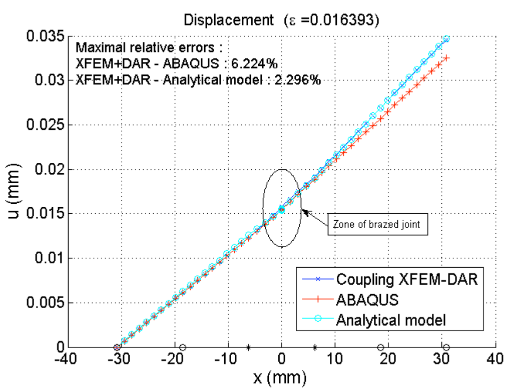

Fig. 4. Solution of the displacement

In this example, there is a change of slope of displacement solution through the brazed joint. The X-FEM/DAR coupled method allows to take this gap into account by using two different exterior enriched functions at the two sides of the brazed joint. The obtained result of displacement by X-FEM/DAR is closely to that by ABAQUS and the analytical computation. The maximal relative errors estimate 6.224% and 2.296% respectively.

6 CONCLUSIONS

A numerical method based on the original coupling X-FEM and DAR was proposed. In the X-FEM, the finite element space is enriched by adding additional functions which are inspired from the solutions of the DAR. This combinaison is implemented in a 1D model of a brazed assembly with elastic deformations. The obtained results show the relevance of this coupling. It promises to be an efficient and robust method which could be more generally applied for any problems implying singularities. However, it is necessary to further improve the accuracy of the solution. Furthermore, we intend to deal the thermal problem. Several routes are currently under investigation in our team.

ACKNOWLEDGEMENTS

The authors are greatful to EMAC research policy for enabling this joined investigation together with M. Nguyen's scholarship.

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