# Discontinuous Galerkin Method for Interface Crack Propagation

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ABSTRACT: In this communication, a class of non-symmetric/symmetric discontinuous Galerkin (dG) methods with interior penalties for interfacial fracture problems is presented. The behaviour of the interface is determined by means of cohesive models depending on the displacement jumps and tractions on the element boundaries. The proposed dG finite element formulation with cohesive models can simplify the computational modeling of failure along well-defined surfaces. Two computational model problems are presented to illustrate the performances of the discontinuous Galerkin method.

Key words: Discontinuous Galerkin method, Interface fracture, Cohesive model

## 1 INTRODUCTION

The interface fracture phenomena play an important role in a number of applications especially in laminated composites. When modelling interface fracture phenomena, the use of discrete approach is advocated to achieve a better representation of the entire fracture process. If failure takes place along well-defined surfaces, a standard way to solve fracture problems with finite element methods consists of inserting interface elements (or cohesive layers) with zero thickness in the mesh at places where cracking is expected to occur [1, 2].

From a modelling perspective, a major drawback of the use of interface elements is that the insertion of cohesive elements introduces an artificial compliance in the structure which is primarily related to the initial slope of the traction-separation law: a stiffer slope introduces a higher initial rigidity between neighbouring bulk elements, thus resulting in a smaller fictitious compliance. On the other hand, a high elastic stiffness of the cohesive surface compared to the elastic stiffness of the bulk material can result in artificial oscillations prior to the opening of the cohesive surface. Moreover, some interface elements require a special topology when they are applied in conjunction with solid-like shell elements [2].

As an alternative method, in this work a non-symmetric/symmetric discontinuous Galerkin (dG) formulation for interfacial crack propagation is presented. The resulting discontinuous Galerkin weak form naturally leads to the implementation of cohesive models. The performances of the dG method for interfacial fracture problems are demonstrated through numerical examples.

## 2 DISCONTINUOUS GALERKIN MATHOD

The discontinuous Galerkin (dG) method is a class of finite element methods, which uses discontinuous, piecewise polynomial spaces for the numerical solution and the test functions [3, 4]. This class of methods is widely used in fluid mechanics, but recently, a significant number of results have been obtained in the area of solid mechanics [3, 4, 5].

The main advantages of the discontinuous Galerkin finite element methods are: the shape functions are discontinuous along the element edges; the dG methods are locally mass conservative at the element level; each element can be thought of as a separate entity - the element topology, the degree of approximation and even the choice of governing equations can vary from element to element and in time over the course of calculation without loss of rigor in the method.

Consider a body that occupies a bounded Lipschitz domain $\Omega$ in $\Re^3$ (see figure 1). The continuum problem is governed by the following equations stated in terms of the Cauchy stress:

$$\begin{aligned} -\nabla \cdot \sigma &= b \quad \text{in } \Omega \\ \sigma \cdot n &= g_N \quad \text{on } \partial_N \Omega \\ u &= g_D \quad \text{on } \partial_D \Omega \end{aligned} \tag{1}$$

where $b$ is the body force, $n$ - the unit vector outward normal to the boundary $\partial\Omega$, $g_D$ and $g_N$ are the boundary conditions applied on the displacement $\partial_D\Omega = \Gamma_D$ and traction $\partial_N\Omega = \Gamma_N$ parts of the boundary, respectively.

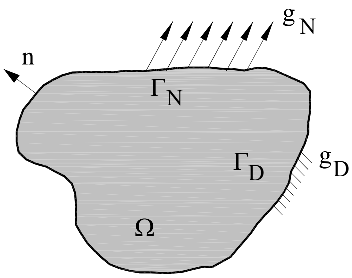


Fig. 1. Body with a discontinuity

Let $\Omega_h = \{T\}$ be a shape-regular partition of $\Omega$, where $T$ are finite elements. Let $e$ denote an arbitrary element edge, and $\varepsilon_h = \{e\}$ be the set of all edges. Each element boundary $e$ is shared by two elements $T^+$ and $T^-$ such that $e = T^+ \cap T^-$, with $n^+$ being the unit normal vector to $T^+$ (see figure 2).

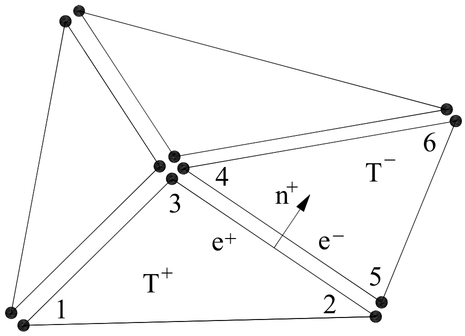


Fig. 2. Discontinuous Galerkin mesh

We decompose $\varepsilon_h$ into three disjoints subsets such that $\varepsilon_h = \varepsilon_I \cup \varepsilon_D \cup \varepsilon_N$, where $\varepsilon_I$ is the set of all internal edges, $\varepsilon_I = \{e \in \partial T \setminus \partial\Omega : T \in \Omega_h\}$; $\varepsilon_D$ - the set of all element edges on the Dirichlet part of the boundary $\partial_D\Omega$, $\varepsilon_D = \{e \subset \partial T \cap \partial_D\Omega : T \in \Omega_h\}$; $\varepsilon_N$ - set of all element edges on the Neumann part of the boundary $\partial_N\Omega$, $\varepsilon_N = \{e \subset \partial T \cap \partial_N\Omega : T \in \Omega_h\}$.

The following approximation space is introduced [4]:

$$V_h = \{v \in L^2(\Omega) : v|_T \in P^k(T), \forall T \in \Omega_h\} \tag{2}$$

where $P^k(T)$ is the space of polynomials of degree at most $k$ supported on $T$.

On interior edges, the definitions of the average and jump are given by

$$\langle x \rangle = \frac{1}{2}\left(x^+ + x^-\right) \ \forall x \in \partial_I\Omega_h, \tag{3}$$

$$[x] = x^+ - x^- \ \forall x \in \partial_I\Omega_h. \tag{4}$$

On outer edges, the jump and the average are $[x] = x^+$, and $\langle x \rangle = x^+$ $\forall x \in \partial_D\Omega_h \cup \partial_N\Omega_h$, respectively. We have assumed that $n^+ = -n^-$ $\forall x \in \partial_I\Omega_h$. The + and – superscripts correspond to evaluating the function at either side of $e$.

To facilitate construction of a numerical scheme with high order accuracy in the vicinity of discontinuities, we require all discontinuities to lie on the element boundaries.

The stabilized discontinuous Galerkin weak formulation results in the following form: Find $u_h \in V_h$ such that:

$$a(v_h, u_h) + j(v_h, u_h) = l(v_h), \ \forall v_h \in V_h \tag{5}$$

in which

$$\begin{aligned} a(v_h,u_h) = &\left(\varepsilon(v_h), \sigma(u_h)\right)_T - \left([v_h], \langle \sigma(u_h)\cdot n \rangle\right)_{\partial_I\Omega_h} + \\ &+ \alpha\left(\langle \sigma(v_h)\cdot n \rangle, [u_h]\right)_{\partial_I\Omega_h} - \left(v_h, \sigma(u_h)\cdot n\right)_{\partial_D\Omega_h} + \\ &+ \alpha\left(\sigma(v_h)\cdot n, u_h\right)_{\partial_D\Omega_h} \end{aligned} \tag{6}$$

$$\begin{aligned} j(v_h,u_h) = &\left(\frac{\beta}{h_e}[v_h],[u_h]\right)_{\partial_I\Omega_h} + \left(\frac{\beta}{h_e}v_h, u_h\right)_{\partial_I\Omega_h} - \\ &- \left(\frac{\beta}{h_e}v_h, g_D\right)_{\partial_D\Omega_h} \end{aligned} \tag{7}$$

and

$$l(v_h) = (v_h, b)_T - \left(v_h, g_N\right)_{\partial_N\Omega_h} + \alpha\left(\sigma(v_h)\cdot n, g_D\right)_{\partial_D\Omega_h} \tag{8}$$

In the above equations $\beta$ is a positive penalty parameter assumed constant across $\Omega_h$, and $h_e$ denotes the characteristic length of the mesh.

The parameter $\alpha$ is either +1, or -1, corresponding to non-symmetric and symmetric interior penalty methods, respectively.

## 3 COHESIVE ZONE MODELING

The cohesive models are a class of models that offer a tool for the investigation of fracture of materials.
In a cohesive model the material separation behavior is described within a softening constitutive equation relating the crack surface tractions to the material separation across the crack surface.
To capture the interfacial crack propagation an irreversible bi-linear, softening cohesive law linking the effective displacement $\delta$ and the effective traction $t$ is employed (see figure 3). The cohesive law is defined in terms of a non-dimensional effective displacement and effective traction defined as [6, 7]:

$$\lambda = \sqrt{(\delta_n/\delta_c)^2 + (\delta_s/\delta_c)^2}\,, \tag{9}$$

and

$$t = \sqrt{t_n^2 + t_s^2} \tag{10}$$

in which $\delta_n$ and $\delta_s$ denote a normal displacement opening and shear sliding, respectively; $\delta_c$ is a critical displacement where complete separation occurs; $t_s$ and $t_n$ are shear and normal tractions, respectively.

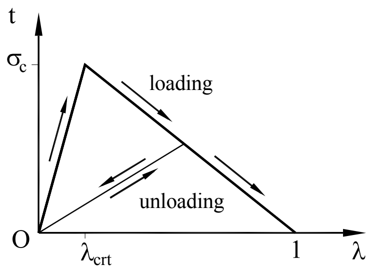


Fig. 3. Bi-linear cohesive law

It is assumed that the traction at the interface increases linearly to its maximum value $\sigma_c$ which corresponds to a displacement $\lambda_{crt}$. Beyond $\lambda_{crt}$, the traction reduce to zero for $\lambda = 1$.
The unloading behavior in the hardening region follows the same slope as the loading path. Reloading follows hardening slope and then continues along the softening slope.
For $0 < \lambda \le \lambda_{crt}$, the shear and normal tractions are defined by

$$t_s = \sigma_c \frac{1}{\lambda_{crt}}\left(\frac{\delta_s}{\delta_c}\right) \text{ and } t_n = \sigma_c \frac{1}{\lambda_{crt}}\left(\frac{\delta_n}{\delta_c}\right) \tag{11}$$

For $\lambda_{crt} < \lambda < 1$, the tractions are defined by

$$t_s = \frac{1-\lambda}{1-\lambda_{crt}}\frac{\sigma_c}{\lambda}\left(\frac{\delta_s}{\delta_c}\right) \text{ and } t_n = \frac{1-\lambda}{1-\lambda_{crt}}\frac{\sigma_c}{\lambda}\left(\frac{\delta_n}{\delta_c}\right) \tag{12}$$

where $\sigma_c$ represents the material strength.

## 4 NUMERICAL EXAMPLES

In order to investigate the effect of integration scheme on traction profile, a simply supported beam is loaded symmetrically by means of an imposed displacement/traction at the centre of the beam on the top edge (see figure 4). The following material properties are used: Young's modulus of 20000 MPa and Poisson's ratio of 0.2.
The simulation was carried out using an object-oriented code written in C++ based on dG formulation presented in Section 2. The mesh consists of 100 four–noded quadrilateral elements in the *x*-direction and 20 elements in *y*-direction.
For the boundaries located at the interface 3–point Gauss/Newton-Cotes integration scheme was used while for the rest of elements 2–point Gauss/Newton-Cotes integration scheme was used.

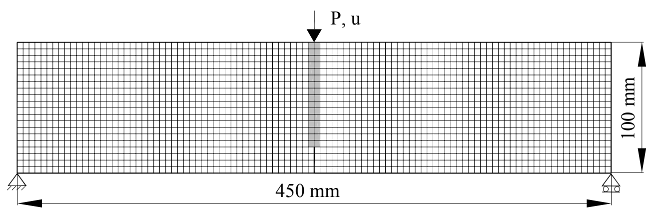


Fig. 4. Geometry of 3-point bending beam

As one can see from figure 5, the dG internal boundaries show no oscillations when a Gauss/Newton-Cotes integration scheme is used for the integration of the traction at the discontinuity.

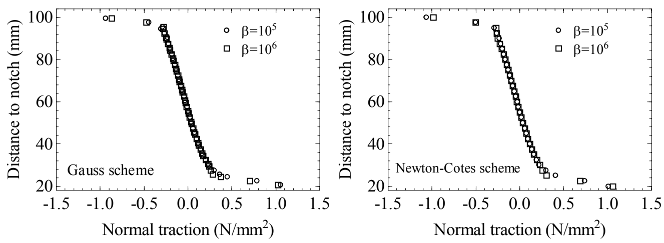


Fig. 5. Effect of integration scheme on traction profiles

It is worthwhile to note that both Gauss and Newton-Cotes integration schemes give a smooth traction profile for different values of the stabilization parameter $\beta$.

To study the capability of the dG method to predict the softening response of a structure given its fracture properties, a compact–tension (CT) specimen is considered (see figure 6). The test is performed under displacement control, with a displacement applied at the uppermost and lowermost left corners of the specimen.

The mesh consists of 2400 four-noded quadrilaterals elements. The last two elements of the interface are prevented from cracking since if a crack propagates through the entire specimen, the system of equations becomes singular as the specimen has no shear resistance and not all rigid body modes are restricted. The initial delamination length of 16.5 mm is modelled as a traction free discontinuity.

The following material properties are used for the simulation: Young's modulus of 100 MPa and a Poisson's ratio of 0.2. The cohesive tensile stress of the material is $\sigma_c$=1.0 MPa while the fracture energy is $G_f$ =0.1 N/mm.

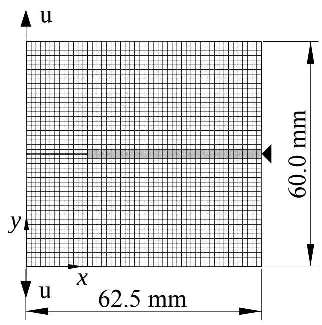


Fig. 6. Geometry of compact test specimen

Due to the symmetry of the geometry, the simulation of pure mode-I crack growth is considered and only normal separation along the interface occurs.

For numerical integration the Newton-Cotes integration scheme is used. For the boundaries located at the interface 3–point Newton Cotes integration scheme was used while for the rest of elements 2–point Newton Cotes integration scheme was employed.

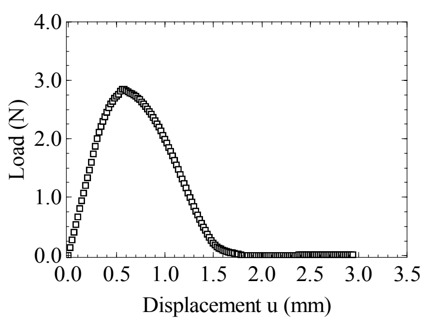


Fig. 7. Load–displacement response

In order to implement the cohesive model within the dG formulation, the tractions on the element boundaries are computed using equation (10). When the traction on the element boundaries exceeds the cohesive tensile stress of the material, a discontinuity is introduced. The load-displacement response is shown in figure 7.

## 5 CONCLUSIONS

The discontinuous Galerkin method can handle cohesive cracks very naturally with some advantages over the other methods, including good stability and consistency, and absence of traction oscillations and spurious reflections. One of the downsides of the dG methods is the computational cost since a loop over the boundaries in the mesh in necessary. Also, an important yet unresolved problem is the automatic selection of the stabilization parameter. However, the proposed dG finite element formulation with cohesive models can simplify the computational modeling of failure along well-defined surface.


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